Module length

Let M be an R-module. Instead of looking at increasing chains of submodules, like w/ Noetherian modules, we consider decreasing chains.

Def: M is Artinian if every strictly decreasing chain of submodules terminates (or ideals in the case of rings).

We'll soon see that all Artinian rings are always Noetherian. In fact, we'll prove something much stronger.

However the converse doesn't necessarily hold:

Ex: $k[x]$ is Noetherian by the Hilbert Basis Theorem. However $(x) \supsetneq (x^2) \supsetneq (x^3) \supsetneq \cdots$ doesn't terminate, so it's not Artinian.

First some definitions:

$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n$ is a (descending) chain of submodules of length n.

The chain is a composition series if each M_j / M_{j+1} is a nonzero simple module (i.e. it has no nonzero proper submodules).

Equivalently, a composition series is a max'l chain of submodules

The length of M, $\ell(M)$, is the least length of a composition series, or ∞ if it has no finite composition series. (In fact, we'll see that all composition series have the same length.)

Remark: Since M_i/M_{i+1} is simple, it is generated by any nonzero element a. So, we get

$$R \twoheadrightarrow M_i/M_{i+1} \quad \text{with kernel}, \qquad 1 \mapsto a$$

$$P = \{r \in R \mid r(M_i/M_{i+1}) = 0\} = \mathrm{Ann}(M_i/M_{i+1}), \text{ so}$$

$M_i/M_{i+1} \cong R/P$. Since the module is simple, P must be maximal.

Lemma: If M has a finite composition series of length n, then every composition series has length n, and every chain in M can be extended to a composition series.

Pf: Let $N \subsetneq M$ be a proper submodule. We first show that $\ell(N) < \ell(M)$.

Let (M_i) be a composition series of M of minimal length. Set $N_i = N \cap M_i$.

Then $N_{i-1}/N_i \subseteq \underset{\substack{\uparrow \\ \text{simple}}}{M_{i-1}/M_i}$, so either N_{i-1}/N_i is simple or $N_{i-1} = N_i$.

Thus, by removing repeated terms, we get a composition series for N, so $\ell(N) \leq \ell(M)$.

If $\ell(N) = \ell(M) = \ell$, then $N_{i-1}/N_i = M_{i-1}/M_i$ for each i, so $M_{\ell-1}/0 = N_{\ell-1}/0 \Rightarrow$ by induction $N = M$, a contradiction.

Now let $M = M_0 \supsetneq M_1 \supsetneq \cdots$ be any chain in M. Then $\ell(M) > \ell(M_1) > \cdots > 0$, so the length of the chain must be $\leq \ell(M) \leq n$. (In particular, $\ell(M) = n$, so all composition series have the same length.)

If a chain has length $\ell(M)$, it thus must be a composition series. Otherwise, it has length $< \ell(M)$ and new terms can be inserted until it has length $\ell(M)$. $\square$

Thm: M has a finite composition series iff M is Artinian and Noetherian.

Pf: Suppose M is Artinian and Noetherian. By ACC, we can find a max'l proper submodule M_1, a max'l submodule M_2 of M_1 etc. By DCC, this terminates in finitely many steps.

If M has a finite composition series, the lemma says every chain of submodules has finite length. $\square$

Now we come back to rings to see how we can interpret Artinianness geometrically.

Thm: Let R be a ring. The following are equivalent:

a.) R is Noetherian and all its prime ideals are maximal.

b.) R is a finite length R-module

c.) R is Artinian.

Pf: a.) $\Rightarrow$ b.) Exercise (see Eisenbud)

b.) $\Rightarrow$ c.) by previous theorem.

c.) $\Rightarrow$ a.) Suppose R is Artinian.

Claim: (0) is a product of max'l ideals.

Pf of Claim: R is Artinian $\Rightarrow$ we can choose an ideal J that is minimal among products of max'l ideals.

Thus, for all max'l $m \subseteq R$, we have $mJ = J$, so $J \subseteq J(R)$ and $J^2 = J$.

Assume $J \neq 0$, and choose I minimal among ideals not annihilating J. Then $(IJ)J = IJ^2 = IJ \neq 0$. So by minimality, $IJ \supseteq I$, and thus $IJ = I$.

Take $f \in I$ s.t. $fJ \neq 0$. By minimality $(f) = I$.

Since $IJ = I$, $\exists\ g \in J$ s.t. $fg = g \Leftrightarrow f(1-g) = 0$. But g is in every max'l ideal, so $1-g$ is in none, so $1-g$ is a unit, so $f = 0$, a contradiction. Thus $J = 0$. $\square$

Thus, we now have $0 = m_1 \cdots m_t$ for m_i max'l ideals.

Set $M_s = m_1 \cdots m_s$. Then we have

$$R \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_t = 0.$$

M_s is an R-module, so $M_s / M_{s+1} = M_s / m_{s+1} M_s$ is an (R/m_{s+1})-vector space. Any descending chain of subspaces corresponds to a chain of ideals in R which is finite, so M_s / M_{s+1} is finite dimensional.

Thus, we can add in finitely many modules to complete this to a finite composition series for R, so R is Noetherian.

Now we just need that all prime ideals are maximal.

Let $P \subseteq R$ prime. Then $P \supseteq m_1 \cdots m_t = 0$. Thus, $P \supseteq m_i$ for some i (otherwise find $a_1 \dots a_t \in P$ s.t. $a_i \notin P$).

Thus P is max'l. $\square$

Note that the last part of the proof implies the following:

Cor: If R is Artinian, then Spec R is finite. (Converse is false: $\mathbb{C}[x]_{(x)}$)

In the case where R is an Artinian k-algebra, we have $R \cong k^{\ell}$ as k-vector space, and ℓ = length of R.

Ex: 1.) $R = \frac{k[x,y]}{(x,y)}$. $\operatorname{Spec} R = \{(0)\}$ and R has length one

$$R \supsetneq (0)$$

Also, $\frac{k[x,y]}{(x,y)} \cong k$.

2.) $R = \frac{k[x]}{(x-1)x}$. $\operatorname{Spec} R = \{(x), (x-1)\}$.

$R = k1 \oplus kx \cong k^2$

Composition series: $R \supsetneq (x) \supsetneq (0)$.

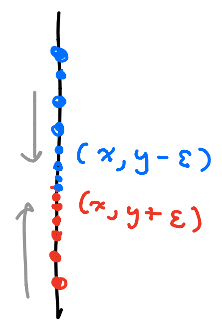
3.) $R = \frac{k[x,y]}{(x, y^2)}$. $\operatorname{Spec} R = \{(x,y)\}$ but $R \cong k1 \oplus ky \cong k^2$

Composition series: $R \supsetneq (y) \supsetneq (0)$.

$\operatorname{Spec} R$ corresponds to a "scheme-y" point:

$\updownarrow$ a point w/ a tangent direction.

Think of this as roughly the limit of two colliding points on a line



We need one more theorem (without proof) that will be useful when we come back to dimension.

Theorem: Let M be an R-module of finite length, $P \subseteq R$ prime. Then $M = M_P \iff M$ is annihilated by a power of P.

Note that we need finite length:

Ex: $M = \mathbb{C}[x,y]/(x^2, xy)$ is a $\mathbb{C}[x,y]$-module. It has infinite length: $M \supsetneq (y) \supsetneq (y^2) \supsetneq \ldots$

Let $P = (x)$. Then $P^2 M = 0$, but $M_P = \mathbb{C}[x,y]_{(x)}/(x) \neq M$.

Cor: Let R be Noetherian, $I \subseteq R$ an ideal. If $P \supseteq I$ is prime then the following are equivalent:

a.) P is minimal among primes containing I.

b.) R_P/I_P is Artinian

c.) $P_P^n \subseteq I_P$ for $n \gg 0$.

Pf: a.) $\Rightarrow$ b.): If P is minimal among primes containing I, then it's the unique prime of R_P/I_P. Thus R_P/I_P is Artinian (since all primes are max'l).

b.) $\Rightarrow$ c.): R_P/I_P Artinian + Noetherian $\Rightarrow \ell(R_P/I_P) < \infty$.

$(R_P/I_P)_P = R_P/I_P$, so by the theorem,

$$P^n \subseteq \mathrm{Ann}_R(R_P/I_P) \quad \text{for } n \gg 0$$

$$\Rightarrow \quad P_P^n \subseteq \mathrm{Ann}_{R_P}(R_P/I_P) = I_P.$$

c.) $\Rightarrow$ a.): Suppose $P_P^n \subseteq I_P$, and let $Q \subseteq R$ a prime ideal s.t. $I \subseteq Q \subseteq P$. Then $P_P^n \subseteq Q_P$. Since Q_P is prime,

$$P_P \subseteq Q_P \Rightarrow P_P = Q_P \Rightarrow P = Q. \ \square$$